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A FEW WORDS ON $M/E_K/1/1$ QUEUING MODEL WITH RENEGING CUSTOMER

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ABSTRACT

In the context of our fast-paced world, customer impatience in queuing systems is a reality. In this paper, we analyze a queuing system with Erlangian service discipline assuming that customers may renege. Analysis in steady state is presented. Relevant performance measures have been discussed.

KEYWORDS: Erlangian service, Finite buffer, Impatience, Queuing, Reneging. 2000 Mathematics Subject Classification: 60K25, 68M20, 90B22.

INTRODUCTION

In the analysis of queues, queuing theorists have model various queuing systems based on their characteristics. Of these various characteristics, one relates to the impatient nature of customer behavior also known as reneging. The phenomenon of customers joining a queuing system and leaving it without service completion is known as reneging. In our modern fast-paced life, customers are hard pressed for time and hence in our day-to-day life reneging can be observed. In spite of the importance of reneging, one does not very often come across papers in queuing literature, which analyzes it.

Reneging can be of two types-viz. reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). A customer can renege only as long as it is in the queue and we call this as reneging of type R_BOS. It cannot renege once it begins receiving service. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service get started i.e. hair cut begins, the customer cannot leave till hair cutting is over. On the other hand, if customers can renege not only while waiting in queue but also while receiving service, we call such behavior as reneging of type R_EOS. An example is processing or merchandising of perishable goods.

In this paper, we analyze customer impatience in $M/E_k/1/1$ model. The model assumes that customers arriving into the queuing system follow Markovian law with rate λ. We also assume single-channel Erlangian service time with rate $k\ \mu$ having k-service stages. We shall further assume that the system capacity is restricted to 1 customer. Since the number of servers is equal to the system capacity, a customer arriving at it either goes straight into the service or is turned away without service (as there is no waiting space) if the server is busy. As waiting is not allowed, balking is not possible in this model. As regards reneging, it is obvious that the reneging rule in the queuing model can only be of R_EOS type. We assume that each customer individually has patience or reneging distribution following exp (ν) in each stage which commences at the instant the customer joins the particular stage. To the best of our knowledge, analysis of reneging in this specific model has not been attempted though some general results for Erlangian service time model can be located in literature. Broadly speaking, very little work analyzing reneging in Erlangian service time models appear to have been done. Shawky (2005) derived the analytical solution of $M/E_r/1/k/N$ for machine interference system with balking and reneging considering FIFO. Some measures of effectiveness and some special cases were obtained. El-Paoumy and Ismail (2009) considered an $M^x/E_k/1/N$ model with balking and reneging. Recurrence relations connecting the various probabilities introduced were calculated. Some measures of effectiveness were deducted and some special cases were also obtained. In both of these papers, closed form expressions of performance measures were not available. This formed the motivation of our work.

The subsequent sections of this paper are structured as follows. Section 2 contains the steady state analysis. In section 3, we discuss some relevant performance measures with closed form expressions. Concluding statements are presented in section 4.

THE SYSTEM STATES ANALYSIS

Let $p_n(t)$ denote the probability that there are 'n' phases in the system at time 't' under R_EOS. Then we can have 0 phase at time t+Δt in the following mutually exclusive ways:

1) 0 phase at time 't', no arrival, no service and no customer leaving the system during next Δt. The probability is

$$p_0(t)\{1-\lambda\Delta t+0(\Delta t)\} \tag{2.1}$$

2) 1 phase at time 't', no arrival, one service and no customer leaving the system during next Δt and another possibility is 1 phase at time 't', no arrival, no service and one customer leaving the system during next Δt . Thus the probability is

$$\begin{aligned}&p_1(t)\{k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\}+p_1(t)\{1-k\mu\Delta t+0(\Delta t)\}\{\nu\Delta t+0(\Delta t)\}\\&=p_1(t)\{k\mu\Delta t+\nu\Delta t+0(\Delta t)\}\end{aligned} \tag{2.2}$$

3) 2 phases at time 't', no arrival, no service and one customer leaving the system during next Δt . The probability is

$$\begin{aligned}&=p_2(t)\{1-k\mu\Delta t+0(\Delta t)\}\{\nu\Delta t+0(\Delta t)\}\\&=p_2(t)\{\nu\Delta t+0(\Delta t)\}\end{aligned} \tag{2.3}$$

And so on. Similarly,

4) k phases at time 't', no arrival, no service and one customer leaving the system during next Δt . The probability is

$$\begin{aligned}&=p_k(t)\{1-k\mu\Delta t+0(\Delta t)\}\{\nu\Delta t+0(\Delta t)\}\\&=p_k(t)\{\nu\Delta t+0(\Delta t)\}\end{aligned} \tag{2.4}$$

From (2.1), (2.2), (2.3) and (2.4) we have,

$$p_0(t+\Delta t)=p_0(t)\{1-\lambda\Delta t+0(\Delta t)\}+p_1(t)\{k\mu\Delta t+\nu\Delta t+0(\Delta t)\}+p_2(t)\{k\mu\Delta t+\nu\Delta t+0(\Delta t)\}+\ldots+ p_k(t)\{k\mu\Delta t+\nu\Delta t+0(\Delta t)\}$$
$$\Rightarrow p_0(t+\Delta t)-p_0(t)=-\lambda\Delta t p_0(t)+k\mu\Delta t p_1(t)+\nu\sum_{i=1}^{k}p_i(t)0(\Delta t)$$

Now dividing both sides of this equation by Δt and taking limit Δt→0, we get

$$p_0'(t)=-\lambda p_0(t)+k\mu p_1(t)+\nu\sum_{i=1}^{k}p_i(t) \qquad (2.5)$$

There can be n phases where 0<n<k, n=1,2,…,k-1 at time t+Δt in the following mutually exclusive ways.

1) There are n phases at time't'; there is no arrival, no service and no customer leaving the system during next Δt. The probability is

$$\begin{aligned}&p_n(t)\{1-k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\}\\ &=p_n(t)\{1-k\mu\Delta t-\lambda(1-p)\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\}\\ &=p_n(t)\{1-\nu\Delta t-k\mu\Delta t+0(\Delta t)\}\end{aligned} \qquad (2.6)$$

2) There are (n+1) phases at time't', no arrival, one service and no customer leaving the system during next Δt. The probability is

$$\begin{aligned}&p_{n+1}(t)\{k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\}\\ &=p_{n+1}(t)\{k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\}\\ &=p_{n+1}(t)\{k\mu\Delta t+0(\Delta t)\}\end{aligned} \qquad (2.7)$$

From (2.6) and (2.7) we have,

$$p_n(t+\Delta t)=p_n(t)\{1-\nu\Delta t-k\mu\Delta t+0(\Delta t)\}+\{k\mu\Delta t+0(\Delta t)\}p_{n+1}(t)$$
$$\Rightarrow p_n(t+\Delta t)-p_n(t)=-\{\nu+k\mu\}\Delta t p_n(t)+k\mu\Delta t p_{n+1}(t)+0(\Delta t)$$

Dividing both sides of this equation by Δt and taking limit Δt→0, we get

$$p_n'(t)=-\{\nu+k\mu\}p_n(t)+k\mu p_{n+1}(t)\text{ ; } 0<n<k,\ n=1,2,\ldots,k \qquad (2.8)$$

There can be k phases at time t+Δt in the following mutually exclusive ways.

1) There are 0 phase at time't'; one arrival, no service and no customer leaving the system during next Δt. The probability is

$$p_0(t).\{\lambda\Delta t+0(\Delta t)\} \qquad (2.9)$$

2) There are k phases at time't', no arrival, no service and no customer leaving the system during next Δt. The probability is

$$\begin{aligned} & p_k(t)\{1-k\mu\Delta t+0(\Delta t)\}\{1-\nu\Delta t+0(\Delta t)\} \\ & = p_k(t)\{1-k\mu\Delta t-\nu\Delta t+0(\Delta t)\} \end{aligned} \tag{2.10}$$

From (2.9) and (2.10) we have,

$$\begin{aligned} & p_k(t+\Delta t)=p_0(t)\{\lambda\Delta t+0(\Delta t)\}+p_k(t)\{1-\nu\Delta t-k\mu\Delta t+0(\Delta t)\} \\ & \Rightarrow p_k(t+\Delta t)-p_k(t)=\lambda p_0(t)-\{\nu+k\mu\}\Delta t p_n(t)+0(\Delta t) \end{aligned}$$

Dividing both sides of this equation by Δt and taking limit $\Delta t \rightarrow 0$, we get

$$p_k'(t)=\lambda p_0(t)-\{\nu+k\mu\}p_n(t) \tag{2.11}$$

Under steady state, the differential equations (2.5), (2.8) and (2.11) become

$$\lambda p_0 = k\mu p_1 + \nu\sum_{i=1}^{k} p_i \tag{2.12}$$

$$(\nu+k\mu)p_n = k\mu p_{n+1} \text{;n=1,2,…,k-1} \tag{2.13}$$

$$(\nu+k\mu)p_k = \lambda p_0 \tag{2.14}$$

From (2.12) we have

$$p_1 = (1/k\mu)(\lambda p_0 - \nu\sum_{i=1}^{k} p_i \tag{2.15}$$

Now multiplying both sides of the equation (2.13) and (2.14) by s^n and summing over the relevant range of n

$$\begin{aligned} & (\nu+k\mu)\sum_{n=1}^{k} p_n z^n = (k\mu/z)\sum_{n=1}^{k} p_{n+1} z^{n+1} + \lambda p_0 z^k \\ & \Rightarrow (\nu+k\mu)\{P(z)-p_0\} = (k\mu/z)\{P(z)-p_0-p_1 z\}+\lambda p_0 z^k \\ & \Rightarrow \{(\nu+k\mu)-(k\mu/z)\}P(z)=\{\nu+k\mu-(k\mu/z)\}p_0+\lambda p_0 z^k - k\mu p_1 \\ & \Rightarrow P(z)=[\{\lambda z(z^k-1)+k\mu(z-1)\}p_0+\nu z]\big/\{z(\nu+k\mu)-k\mu\} \end{aligned}$$

Using (2.15) Then,

$$P'(1)=\{(\lambda+\mu)kp_0-k\mu\}/\nu \tag{2.16}$$

From (2.12), (2.13) and (2.14) we can determine the probability that there are 'n' phases in the system and it is given by

$$p_n = \frac{\lambda (k\mu)^{k-n}}{(\nu + k\mu)^{k-n+1}} p_0 \text{ ; n=1,2,…,k}$$

where

$$p_0 = \left[1 + \sum_{n=1}^{k} \frac{\lambda(k\mu)^{k-n}}{(\nu + k\mu)^{k-n+1}}\right]^{-1} \tag{2.17}$$

PERFORMANCE MEASURES

In general, "performance measures are the specific representation of a capacity, process or outcome deemed relevant to the assessment of performance, which are quantifiable and can be documented" (www.iphionline.org). The main objective of any queuing study is to assess some well-defined parameters, which are designed at striking a good balance between customer satisfaction and economic considerations. In queuing theory, measures through which the nature of the quality of service can be studied are known as performance measures. Performance measures are important as the analysis of relevant performance measures of queuing models allows the cause of queuing issues to be identified and the impact of proposed system changes to be assessed. Some of the performance measures of any queuing system that are of general interest for the evaluation of the performance of an existing queuing system and to design a new system in terms of the level of service a customer receives as well as the proper utilization of the service facilities include mean size, server utilization, customer loss and the like.

An important measure is the mean number of customers in the system, which is traditionally denoted by 'L'. From (2.16), the mean system size is given by

$$L = \{(\lambda + \mu) k p_0 - k\mu\} / \nu$$

Customers arrive into the system at the rate of λ. However all the customers who arrive do not join the system because of finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\begin{aligned} \lambda^e &= \lambda \sum_{n=0}^{k-1} p_n \\ &= \lambda (1 - p_k) \end{aligned} \tag{3.1}$$

where

$$p_k = \frac{\lambda}{(k\mu + \nu)} p_0$$ 'p_0' is given in (2.17)

We have assumed that each customer has a random patience time following exp (ν). Clearly then, the reneging rate of the system would depend on the state of the system. The average reneging rate (avg rr) is given by

$$Avg\ \ rr = \sum_{n=1}^{k} \nu p_n = \nu(1 - p_0) \tag{3.2}$$

In system management, customers who renege represent business lost. It is therefore of interest to determine the proportion of customers lost, both out of these joining the system as well as out of those arriving into the system. These are given below

Proportion of customer lost due to reneging out of those arriving and joining the system is

$$= Avgrr\ /\lambda^e$$

$$= \nu(1 - p_0)/\lambda(1 - p_k)$$
$$= \{(\nu + k\mu)(1 - p_0)\}/(\nu + k\mu - \lambda p_0)$$

Proportion of customer lost due to reneging out of total customers arriving in the system is

$$= Avgrr\ /\lambda$$
$$= \nu(1 - p_0)/\lambda$$

In totality, customers are lost to the system in two ways, due to finite buffer and due to reneging. The management would like to know the proportion of total customers lost in order to have an idea of total business lost. Hence the mean rate at which customers are lost is

Rate of loss due to finite buffer+ Avgrr

$$= \lambda - \lambda^e + Avgrr$$

$$= (\lambda/\nu)(1 - p_0) + p_k$$
$$= (\lambda/\nu)\{1 - (k\mu p_0)/(\nu + k\mu)$$

This rate helps in the determination of proportion of customers lost which is

$$= \{\lambda - \lambda^e + avgrr\}/\lambda$$

$$= (1/\nu)\{1 - (k\mu p_0)/(\nu + k\mu)$$

The proportion of customers completing service is its complement.

CONCLUSIONS

The analysis of queuing system with Erlangian service time and Markovian reneging with finite buffer has been presented. Analysis in steady state has been discussed. Even though reneging in this model have been discussed by others, explicit closed form expression of performance measures are not available. This paper makes a contribution here. The limitations of this work stem from the fact that the system capacity is restricted to one customer only. Extension of our results for system capacity greater than one is a pointer to future research.

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